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New application of the the large-N_c expansion: comparison of the Gottfried and Adler sum rules

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ABSTRACT

The Adler sum rule for deep inelastic neutrino scattering measures the isospin of the nucleon, and is hence exact. In contrast the Gottfried sum rule for charged lepton scattering does receive perturbative and non-perturbative corrections. We show that at two-loop level the Gottfried sum rule is suppressed by a factor $1/N_c^2$ relative to higher moments, and we conjecture that this suppression holds to all-orders, and also for higher-twist effects. It is further noted that the *differences* between radiative corrections for higher moments of neutrino and charged lepton deep inelastic scattering, are $1/N_c^2$ suppressed at two-loops, and this is also conjectured to hold to all-orders. The $1/N_c^2$ suppression of perturbative corrections to the Gottfried sum rule makes it plausible that the deviations from the parton model value are dominated by a light quark flavour asymmetry in the nucleon sea. This asymmetry indeed persists as $N_c \to \infty$ as predicted in a chiral-soliton model.

In this talk we describe the results of the recent work of Ref. [1], in which we are led to a conjecture concerning the radiative QCD corrections (both perturbative and higher-twist) to non-singlet neutrino nucleon and charged nucleon Deep Inelastic Scattering (DIS). Let us begin by considering the isospin Adler sum rule, which is the first non-singlet moment for neutrino DIS. This has the parton model expression

$$\begin{aligned} I_A &\equiv \int_0^1 \frac{dx}{x}\left[F_2^{\nu p}(x,Q^2) - F_2^{\nu n}(x,Q^2)\right] \\ &= 2\int_0^1 dx\left(u(x) - d(x) - \bar{u}(x) + \bar{d}(x)\right) \\ &= 4I_3 = 2\ . \end{aligned} \tag{1}$$

Since isospin is conserved this sum rule has the special feauture that it is exact, and receives no perturbative or non-perturbative QCD corrections. This expectation of $I_A = 2$ is consistent with existing neutrino-nucleon DIS [2], which show no significant Q^2 variation

in the range $2\text{GeV}^2 \leq Q^2 \leq 30\text{GeV}^2$ and give

$$I_A^{\text{exp}} = 2.02 \pm 0.40 \; . \tag{2}$$

The corresponding sum rule for charged-lepton-nucleon DIS has the form

$$\begin{aligned} I_G(Q^2) &= \int_0^1 \frac{dx}{x} \left[F_2^{lp}(x,Q^2) - F_2^{ln}(x,Q^2) \right] \\ &= \frac{1}{3} \int_0^1 dx \left(u(x) - d(x) + \bar{u}(x) - \bar{d}(x) \right) \\ &= \frac{1}{3} - \frac{2}{3} \int_0^1 dx \left(\bar{d}(x) - \bar{u}(x) \right) \; . \end{aligned} \tag{3}$$

If the nucleon sea were flavour symmetric with $\bar{u}(x) = \bar{d}(x)$ then one has the valence contribution to the Gottfried sum rule $I_G^v = 1/3$ only. This value strongly *disagrees* with the data as analysed by the NMC collaboration [3] which gave the following result

$$I_G^{\text{exp}}(Q^2 = 4 \text{ GeV}^2) = 0.235 \pm 0.026 \; . \tag{4}$$

In contrast to the Adler sum rule the Gottfried sum rule is not exact and will be modified by both perturbative and non-perturbative corrections. The perturbative corrections to $\text{O}(\alpha_s^2)$ were analysed numerically in Ref. [4] and were found to be small. They cannot explain the discrepancy between the NMC data and the naive expectation $I_G^v = 1/3$. A possible resolution is the existence of a light quark flavour asymmetry with $\bar{u}(x,Q^2) < \bar{d}(x,Q^2)$ over a significant x-range.

In the case of flavour symmetric sea the perturbative QCD corrections to the Gottfried sum rule can be written in the form

$$I_G^v(Q^2) = A(\alpha_s) C^{(l)}(\alpha_s) \; , \tag{5}$$

where A is the anomalous dimension contribution and $C^{(l)}$ is the coefficient function.

$$C^{(l)}(\alpha_s) = \frac{1}{3} \left[1 + \sum_{n=1}^{\infty} C_n^{(l)N=1} \left(\frac{\alpha_s}{\pi} \right)^n \right] \; . \tag{6}$$

The coefficient function receives no corrections at $\text{O}(\alpha_s)$, and numerical integration of the two-loop results of Van Neerven and Zijlstra [5] gave [4]

$$C_2^{(l)N=1} = (3.695 C_F^2 - 1.847 C_F C_A) \; , \tag{7}$$

where $C_A = N_c$ and $C_F = (N_c^2 - 1)/2N_c$ are QCD Casimirs. Combining with the anomalous dimension part then yields for $N_F = 3$ quark flavours

$$I_G^v(Q^2) = \frac{1}{3} \left[1 + 0.0355 \left(\frac{\alpha_s}{\pi} \right) + \left(-0.853 + \frac{\gamma_2^{N=1}}{64\beta_0} \right) \left(\frac{\alpha_s}{\pi} \right)^2 \right] \; . \tag{8}$$

Here $\gamma_2^{N=1}$ is the three-loop anomalous dimension coefficient for e $I_G^v(Q^2)$, which at the time of the calculation of [4] was unknown. The relative one-loop anomalous dimension coefficient is zero, and at two-loops the result of calculations [6] is:

$$\gamma_1^{N=1} = -4(C_F^2 - C_F C_A/2)[13 + 8\zeta(3) - 12\zeta(2)] \; . \tag{9}$$

It is noteworthy that this is proportional to the typical non-planar colour factor $(C_F^2 - C_F C_A/2)$, which is $\mathrm{O}(1/N_c^2)$ suppressed relative to the individual weights C_F^2 and $C_F C_A$. For higher moments $N > 1$ this cancellation does not occur. The formulation of the conjecture started with our observing that the numerically calculated two-loop coefficient of Eq.(7) can be rewritten in the form

$$\begin{aligned} C_2^{(l)N=1} &= (3.695 C_F^2 - 1.847 C_F C_A) \\ &= 3.695(C_F^2 - C_F C_A/2.0005) \;. \end{aligned} \tag{10}$$

So to four sugnificant figures the non-planar colour factor is reproduced. This suggests the conjecture that in fact the perturbative corrections to the Gottfried sum rule are purely non-planar and are suppressed in the large-N_c limit.

At two-loops one can show that this is indeed the case. The two-loop coefficient $C_2^{(l)N=1}$ can be defined through the $N=1$ Mellin moment of NS lepton-nucleon DIS

$$C_2^{(l)N} = 3\int_0^1 dx[C^{(2),(+)}(x,1) + C^{(2),(-)}(x,1)]x^{N-1} \;. \tag{11}$$

The two-loop functions $C^{(2),(+)}$ and $C^{(2),(-)}$ have been computed by van Neerven and Zijlstra [5], and confirmed using another technique by Moch and Vermaseren [7]. For neutrinoproduction DIS the corresponding moments involve these *same* functions,

$$C_2^{(\nu)N} = \frac{1}{2}\int_0^1 dx[C^{(2),(+)}(x,1) - C^{(2),(-)}(x,1)]x^{N-1} \;. \tag{12}$$

The $N=1$ case corresponds to the Adler sum rule which has vanishing corrections, so that $C_2^{(\nu)N=1} = 0$, and we can conclude that

$$\int_0^1 dx C^{(2),(+)}(x,1) = \int_0^1 dx C^{(2),(-)}(x,1) \;. \tag{13}$$

We can then use this relation to eliminate $C^{(2),(+)}$ from $C_2^{(l)N=1}$ to obtain

$$C_2^{(l)N=1} = 2\times 3\int_0^1 dx C^{(2),(-)}(x,1) \;. \tag{14}$$

Thus the Gottfried sum rule perturbative coefficients only involve the $C^{(2),(-)}$ function. One can check directly from the explicit results of [5] that whilst $C^{(2),(+)}$ receives both planar and non-planar contributions, $C^{(2),(-)}$ is explicitly proportional to the non-planar factor $(C_F^2 - C_F C_A/2)$. Performing the $C^{(2),(-)}$ integration to thirty significant figures using MAPLE, and matching to the expected structures $\{1, \zeta_2, \zeta_3, \zeta_4\}$, gives an analytical formula for the two-loop coefficient,

$$C_2^{(l)N=1} = \left[-\frac{141}{32} + \frac{21}{4}\zeta(2) - \frac{45}{4}\zeta(3) + 12\zeta(4)\right] C_F(C_F - C_A/2) \;. \tag{15}$$

To finally show that to $\mathrm{O}(\alpha_s^2)$ the perturbative corrections to the Gottfried sum rule are suppressed in the large-N_c limit one needs to compute $\gamma_2^{N=1}$. The recent calculation of

three-loop non-singlet splitting functions by Moch, Vermaseren and Vogt [8] enabled us to compute this with the result

$$
\begin{aligned}
\gamma_2^{N=1} &= (C_F^2 - C_A C_F/2)\Big\{ C_F\Big[290 - 248\zeta(2) + 656\zeta(3) \\
&\quad - 1488\zeta(4) + 832\zeta(5) + 192\zeta(2)\zeta(3)\Big] \\
&\quad + C_A\Big[\frac{1081}{9} + \frac{980}{3}\zeta(2) - \frac{12856}{9}\zeta(3) \\
&\quad + \frac{4232}{3}\zeta(4) - 448\zeta(5) - 192\zeta(2)\zeta(3)\Big] \\
&\quad + N_F\Big[-\frac{304}{9} - \frac{176}{3}\zeta(2) + \frac{1792}{9}\zeta(3) - \frac{272}{3}\zeta(4)\Big]\Big\} \\
&\approx 161.713785 - 2.429260\, N_F
\end{aligned}
\tag{16}
$$

which was obtained using the results of [9]. There is indeed an overall non-planar colour factor.

One can extend the conjecture to higher moments $N > 1$. $C^{(l)N}$ and $C^{(\nu)N}$ both contain the *same* $C^{(2),(+)}$ term, and have an opposite sign $C^{(2),(-)}$ term. This immediately implies that at two-loops the coefficient functions for higher moments have identical planar contributions and *differ* by non-planar contributions which are suppressed in the large-N_c limit. The anomalous dimension coefficients for general moments of non-singlet lepton-nucleon and neutrino-nuleon DIS, $\gamma_n^{(l)N}$ and $\gamma_n^{(\nu)N}$, can be related to splitting functions $P^{(n)+}(x)$ and $P^{(n)-}(x)$, [6, 7]

$$
\gamma_n^{(l)N} = -2\int_0^1 dx P^{(n)+}(x)x^{N-1} \tag{17}
$$

and

$$
\gamma_n^{(\nu)N} = -2\int_0^1 dx P^{(n)-}(x)x^{N-1}\ . \tag{18}
$$

At both two-loops [6] and at three-loops [7] one can check that the difference of $(+)$ and $(-)$ splitting functions, $P^{(n)+}(x) - P^{(n)-}(x)$, is proportional to $C_F(C_F - C_A/2)$ and is non-planar. This immediately demonstrates, using Eqs.(17) and (18) above, that up to and including three-loop order the anomalous dimension coefficients $\gamma_n^{(l)N}$ and $\gamma_n^{(\nu)N}$ *differ* by non-planar terms, supressed in the large-N_c limit. Notice that the $N = 1$ moment is again a special case since the vanishing corrections to the Adler sum rule require that $\gamma_n^{(\nu)N=1} = 0$. From Eq.(17) this implies that $\int_0^1 dx P^{(n)-}(x) = 0$, and hence given that the difference of splitting functions is non-planar and for $n = 1, 2$ $\gamma_n^{(l)N=1}$ in Eqs. (9),(16) for the Gottfried sum rule is also non-planar. The conjecture is that these features persist at all-loops. We can formulate the conjecture in a more precise way by introducing the "planar approximation" [10].

The *planar approximation* retains only those terms at $\mathrm{O}(\alpha_s^n)$ which contain the leading

N_c behaviour for each possible power of N_F. That is we define

$$C_n^N|_{\rm planar} = C_F \sum_{i=0}^{n-1} \mathcal{C}_{n,i}^N N_F^{n-1-i} N_c^i \ , \tag{19}$$

where the $\mathcal{C}_{n,i}^N$ are pure numbers. The above conjectures then amount to the statement that

$$\mathcal{C}_{n,i}^{(l)N} = 6\mathcal{C}_{n,i}^{(\nu)N} \ , \tag{20}$$

or equivalently that

$$C_n^{(l)N}|_{\rm planar} = 6C_n^{(\nu)N}|_{\rm planar} \ , \tag{21}$$

for all moments and for all-loops. The relative factor of 6 simply reflects the relative normalisation of the parton model sum rules. We should note that the neutrino-nucleon moments will involve quark-loop terms involving $N_F d^{abc}d_{abc}/N_c$, and we are assuming that such terms are discarded. The $N = 1$ case is special precisely because the Adler sum rule is exact, which ensures that $C_n^{(l)N=1}|_{\rm planar} = 0$ for the Gottfried sum rule.

If indeed the perturbative corrections to the Gottfried sum rule are non-planar then this suggests that the infrared renormalons associated with its coefficient function are also suppressed in the large-N_c limit, but since IR renormalon ambiguities are of the same form as OPE higher-twist power corrections, this implies that such higher-twist corrections are also suppressed in the large-N_c limit. This raises the possibility that the dominant corrections to I_G^v arise from a light quark flavour asymmetry. One way of modelling this non-perturbative effect is the chiral soliton model, which has been used in Ref. [11] to estimate corrections to the Gottfried sum rule. In this model one finds

$$\frac{1}{2}(3I_G^v - 1) = \int_0^1 dx \left(\bar{u}(x) - \bar{d}(x)\right) = O(N_c^0). \tag{22}$$

So the flavour asymmetry *persists* in the large-N_c limit. These authors obtained an estimate consistent with I_G in the range 0.219 to 0.178, in fair agreement with $I_G^{\rm exp}$ = 0.235±0.026.

Whilst we believe that there is compelling evidence that our general conjectures are correct, what is clearly lacking is a proof. A further check will be possible when the functions $C^{(3),(+)}(x)$ and $C^{(3),(-)}(x)$ are known, work on this is underway at present [12]. The conjecture could then be confirmed up to and including three-loops.

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